A QUOTIENT RESTRICTION THEOREM FOR ACTIONS OF REAL REDUCTIVE GROUPS

HENRIK STÖTZEL

Abstract. We prove a version of the Chevalley Restriction Theorem for the action of a real reductive group G on a topological space X which locally embeds into a holomorphic representation. Assuming that there exists an appropriate quotient $X/\!\!/G$ for the G-action, we introduce a stratification which is defined with respect to orbit types of closed orbits. Our main result is a description of the quotient $X/\!\!/G$ in terms of quotients by normalizer subgroups associated to the stratification.

1. Introduction

Let G be a connected complex semisimple Lie group and let us consider the adjoint representation of G on its Lie algebra $\mathfrak{g}$. If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, then the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ induces a homomorphism of the algebra $\mathbb{C}[\mathfrak{g}]$ of polynomials on $\mathfrak{g}$ into the algebra $\mathbb{C}[\mathfrak{h}]$ of polynomials on $\mathfrak{h}$. Let $\mathbb{C}[\mathfrak{g}]^G$ denote the set of invariant polynomials on $\mathfrak{g}$ with respect to the adjoint action of G on $\mathfrak{g}$. If $\mathcal{W}$ is the Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{h}$, we denote by $\mathbb{C}[\mathfrak{h}]^{\mathcal{W}}$ the set of $\mathcal{W}$-invariant polynomials on $\mathfrak{h}$. The Chevalley restriction theorem states

(Chevalley) *The inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ induces an isomorphism $\mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}[\mathfrak{h}]^{\mathcal{W}}$.*

In terms of algebraic quotients this means that we have a canonical isomorphism $\mathfrak{h}/\!\!/\mathcal{W} \to \mathfrak{g}/\!\!/G$ of algebraic varieties.

The Chevalley restriction theorem was generalized by Luna and Richardson ([LR79]) and by Schwarz ([Sch80]) to actions of complex reductive groups on affine varieties and to actions of compact groups on smooth manifolds, respectively.

Assume that X is an irreducible normal affine variety equipped with a regular action of a complex reductive group and denote by $\pi\colon X \to X/\!\!/G$ the algebraic quotient. Then there exists a Zariski open subset U of X and a subgroup H of G such that each fiber of the restricted quotient $U \to U/\!\!/G$ contains a closed orbit of orbit type G/H. Let X^H be the set of H-fixed points in X and let $\mathcal{N}_G(H)$ be the normalizer of H in G.

(Luna, Richardson) *Assume that the quotient $X^H/\!\!/\mathcal{N}_G(H)$ is irreducible. Then the inclusion $X^H \hookrightarrow X$ induces an isomorphism $X^H/\!\!/\mathcal{N}_G(H) \to X/\!\!/G$ of affine varieties.*

Let G be compact and let X be a smooth G-manifold such that the quotient X/G is connected. Then there exists a generic isotropy group, i. e. a subgroup H of G such that in a G-invariant dense open subset of X each orbit is of orbit type G/H. Let $X^{<H>} := \{x \in X; G_x = H\}$ and let $\overline{X^{<H>}}$ denote the closure of $X^{<H>}$.

(Schwarz) *The inclusion $\overline{X^{<H>}} \hookrightarrow X$ induces a homeomorphism $\overline{X^{<H>}}/\mathcal{N}_G(H) \to X/G$.*

We give a version of the Chevalley Restriction Theorem for actions of real reductive groups on topological spaces which are locally embedded into representations.

1991 *Mathematics Subject Classification.* 57S20, 53D20, 22E46.
The author is supported by the Sonderforschungsbereich SFB/TR12 of the Deutsche Forschungsgemeinschaft.

More precisely our setup is as follows. Let $U^{\mathbb{C}}$ be a complex reductive group with compact real form U. Then the map $U \times i\mathfrak{u} \to U^{\mathbb{C}}$, $(u, \xi) \mapsto u \exp(\xi)$, is a diffeomorphism. We call a closed subgroup G of $U^{\mathbb{C}}$ real reductive, if $G = K \exp(\mathfrak{p})$ for $K := G \cap U$ and $\mathfrak{p} := \mathfrak{g} \cap i\mathfrak{u}$.

If X is a G-representation which is given by restriction of a finite dimensional holomorphic $U^{\mathbb{C}}$-representation, then the topological Hilbert quotient $\pi\colon X \to X/\!\!/G$ exists. By definition, this quotient identifies two points in X if and only if the closures of the G-orbits through these points intersect. For a subset $Y \subset X$ which is G-saturated, i. e. for which $\pi^{-1}(\pi(Y)) = Y$ holds, or which is closed and G-invariant, the quotient $Y \to Y/\!\!/G$ exists and is given by the restriction $\pi|Y\colon Y \to \pi(Y)$. We call a topological G-space X a locally G-semistable space, if the topological Hilbert quotient $\pi\colon X \to X/\!\!/G$ exists, and if for each $x \in X$ there exists a G-saturated open neighborhood W of x and a complex reductive group $U^{\mathbb{C}}$ containing G as a closed compatible subgroup such that W is G-equivariantly homeomorphic to a closed G-invariant subset of an open G-saturated subset of a holomorphic $U^{\mathbb{C}}$-representation space V.

If X is a locally G-semistable space and $G{\cdot}x$ is a closed orbit in X, then there exists a geometric G-slice at x as follows. Let G_x be the G-isotropy group at x and let S be a G_x-invariant locally closed subset of X which contains x. Then G_x acts on $G \times S$ by $h \cdot (g, s) = (gh^{-1}, h \cdot s)$ and the restricted action $G \times S \to X$, $(g, s) \mapsto g \cdot s$ induces a map $\Psi\colon G \times^{G_x} S \to GS$ where $G \times^{G_x} S$ denotes the quotient $(G \times S)/G_x$. We call S a geometric G-slice, if Ψ is a homeomorphism onto an open subset. Here it follows from the construction that the slice S can be chosen such that GS is G-saturated and S is a locally G_x-semistable space.

For a locally G-semistable space X, each fiber of the quotient $\pi\colon X \to X/\!\!/G$ contains a unique closed orbit and this orbit is also the unique orbit of minimal dimension in that fiber. For a compatible subgroup H of G, we define $X^{<H>}$ to be the set of points $x \in X$ such that $G \cdot x$ is closed and $G_x = H$. We call $I_H(X) := \pi^{-1}(\pi(X^{<H>}))$ the G-isotropy stratum of H in X. One of our results is that the sets $I_H(X)$ define a stratification of X.

If G is complex reductive and acts regularly on an irreducible complex space, then there exists a dense stratum. This is also the case if G is compact and X is a smooth G-manifold such that the quotient X/G is connected. For actions of real reductive groups, the stratification is more delicate. Even if X is a G-representation space, a dense stratum does not necessarily exist. Originally we were interested in actions on smooth manifolds but the fact that strata are not smooth in general motivated our definition of a locally G-semistable space. Here a stratum in X and even the closure of a stratum are again locally G-semistable spaces and they contain a dense stratum.

In the following, we assume that the locally G-semistable space X contains a dense stratum. Let $x_0 \in X$ and let $G \cdot x$ be the unique closed orbit in the fiber $\pi^{-1}(\pi(x_0))$. Let $G \times^{G_x} S \to GS$ be a slice at x. Since S is a locally G_x-semistable space, we have the notion of G_x-isotropy strata in S. We define $n(x_0)$ to be the number of open G_x-isotropy strata in S which contain x in their closure and we call $n(x_0)$ the splitting number at x_0. Note that the splitting number is one in the simple cases where the existence of a dense stratum is guaranteed.

Our main result is the following.

Restriction Theorem. *Let X be a locally G-semistable space containing an open and dense G-isotropy stratum $I_H(X)$. Then the topological Hilbert quotient $\pi_N\colon \overline{X^{<H>}} \to \overline{X^{<H>}}/\!\!/\mathcal{N}_G(H)$ exists and the inclusion $\overline{X^{<H>}} \hookrightarrow X$ induces a continuous finite surjective map*

$$\Phi\colon \overline{X^{<H>}}/\!\!/\mathcal{N}_G(H) \to X/\!\!/G.$$

For $x \in X$, the number of points in the fiber $\Phi^{-1}(\pi(x))$ is equal to the splitting number $n(x)$. If $x \in \overline{X^{<H>}}$ with $n(x) > 1$, then Φ is not open at $\pi_N(x)$.

Here by a finite map, we mean a proper map with finite fibers. In the special case where the splitting number is constantly one, the map Φ is a homeomorphism. The splitting number is one for points in the stratum $I_H(X)$ and $I_H(X)$ is again a locally G-semistable space, so in particular the restriction $X^{<H>}/\!\!/\mathcal{N}_G(H) \to I_H(X)/\!\!/G$ of Φ is a homeomorphism.

Our result is new even for a representation of a semisimple real group or more generally if X is a G-representation space which is given by restriction of a holomorphic $U^{\mathbb{C}}$-representation. If the representation space X contains a dense stratum, then $\overline{X^{<H>}}$ is the subspace of H-fixed points in X. In particular $\overline{X^{<H>}}$ is smooth. This holds also if X is a smooth locally G-semistable space containing a dense stratum.

If G is complex reductive and if X is a holomorphic G-representation, a dense stratum always exists and the splitting number is constantly one. Consider the adjoint representation of the complex reductive group G on its Lie algebra $X = \mathfrak{g}$. Here $\overline{X^{<H>}}$ is a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. The group H acts trivially on $\mathfrak{h}$, so the quotient $\overline{X^{<H>}}/\!\!/\mathcal{N}_G(H)$ coincides with the geometric quotient $\mathfrak{h}/\mathcal{W}$ where $\mathcal{W} = \mathcal{N}_G(H)/H$ is the Weyl group. Thus we get the assertion of the Chevalley restriction theorem. For the adjoint representation of a real reductive group, a dense stratum does not necessarily exist. But for each open stratum $I_H(\mathfrak{g})$, the Lie algebra $\mathfrak{h}$ is a real Cartan subalgebra. So, restricting the quotient $\mathfrak{g}/\!\!/G$ to $I_H(\mathfrak{g})$ we get a version of the Chevalley restriction theorem for the adjoint representation of a real reductive group.

If X is an irreducible normal variety equipped with a regular action of a complex reductive group G then a dense stratum exists and the splitting number is constantly one. Due to normality of X, the map Φ is an isomorphism of varieties. If the quotient $X^H/\!\!/\mathcal{N}_G(H)$ is irreducible then it coincides with $\overline{X^{<H>}}/\!\!/\mathcal{N}_G(H)$ and we obtain the theorem of Luna and Richardson where it is assumed that $X^H/\!\!/\mathcal{N}_G(H)$ is irreducible.

If G is compact and X is smooth, then the condition that X/G is connected guarantees that there exists a dense stratum in X. Moreover the splitting number is constantly one, so we obtain the result of Schwarz.

The author would like to thank P. Heinzner and G. Schwarz for helpful discussions and remarks on the content of this paper.

2. Locally semistable spaces

2.1. **Real reductive groups.** Let $U^{\mathbb{C}}$ be a complex reductive Lie group with compact real form U. If $\mathfrak{u}$ denotes the Lie algebra of U, then $U^{\mathbb{C}} = U \exp(i\mathfrak{u})$. Here the map $U \times i\mathfrak{u} \to U^{\mathbb{C}}$, $(u, \xi) \mapsto u\exp(\xi)$ is a diffeomorphism. Note that $U^{\mathbb{C}}$ is the universal complexification of U in the sense of [Ho65]. We denote by θ the Cartan involution with fixed point set U.

We say that a Lie subgroup G of $U^{\mathbb{C}}$ is *compatible*, if $G = K\exp(\mathfrak{p})$ for a subgroup K of U and a subspace $\mathfrak{p}$ of $i\mathfrak{u}$. A θ-stable closed subgroup of $U^{\mathbb{C}}$ is compatible if and only if it has only finitely many connected components (see e.g. Lemma 1.1.3 in [Mie07]). We call $G = K\exp(\mathfrak{p})$ the Cartan decomposition of G. Note that K is compact if and only if G is closed. Moreover, we call a Lie subgroup H of G *compatible* if it is compatible with the Cartan decomposition of $U^{\mathbb{C}}$ or equivalently, if there exists a subgroup L of K and a subspace $\mathfrak{q}$ of $\mathfrak{p}$ such that $H = L\exp(\mathfrak{q})$. Note that the latter condition depends only on the Cartan decomposition of G and not on the choice of $U^{\mathbb{C}}$. In the rest of this paper, G will denote a closed compatible subgroup of a fixed complex reductive group $U^{\mathbb{C}}_G$ and $G = K\exp(\mathfrak{p})$ will denote the associated Cartan decomposition of G.

2.2. **The topological Hilbert quotient.** Let X be a topological G-space, i. e. a topological space equipped with a continuous action of G. We define a relation on X by setting $x \sim y$ if and only if the closures $\overline{G\cdot x}$ and $\overline{G\cdot y}$ of the orbits $G\cdot x$ and $G\cdot y$ intersect. If this relation is an equivalence relation, we define $X/\!\!/G := X/\sim$ and call the quotient $\pi\colon X \to X/\!\!/G$ the

topological Hilbert quotient. If every G-orbit in X is closed, in particular if G acts properly on X, then the quotient $X/\!\!/G$ is the usual orbit space X/G. This happens automatically if G is compact.

Assume that the topological Hilbert quotient $X \to X/\!\!/G$ exists. A subset $Y \subset X$ is called *G-saturated*, if $Y = \pi^{-1}(\pi(Y))$ holds, or equivalently, if $x \in Y$ whenever there exists a $y \in Y$ with $\overline{G \cdot x} \cap \overline{G \cdot y} \neq \emptyset$. We say that a subset Y of X is *G-open* if it is open and G-saturated. Furthermore, we call a subset Y of X a *G-locally closed subset*, if the following equivalent conditions are satisfied.

- Y is a G-invariant closed subset of a G-open subset of X.
- Y is the intersection of a G-open and a closed G-invariant subset of X.
- Y is G-invariant, locally closed, and an orbit $G \cdot y \subset Y$ is closed in Y if and only if it is closed in X.

For a G-locally closed subset Y of X the quotient $Y \to Y/\!\!/G$ exists and is obtained by restriction of the quotient $X \to X/\!\!/G$.

2.3. **Semistable spaces.** Recall that G is a compatible subgroup of a complex reductive group $(U_G)^{\mathbb{C}}$ with Cartan involution θ. Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation of G on a finite dimensional complex vector space V such that $\rho \circ \theta = \theta' \circ \rho$ for a Cartan involution θ' of $\mathrm{GL}(V)$. Equivalently, we assume that we are given a complex reductive group $U^{\mathbb{C}}$ which contains G as a compatible subgroup and a holomorphic representation $\rho\colon U^{\mathbb{C}} \to \mathrm{GL}(V)$. For this, note that $U^{\mathbb{C}} := (U_G)^{\mathbb{C}} \times \mathrm{GL}(V)$ is complex reductive and that the map $g \mapsto (g, \rho(g))$ embeds G into $U^{\mathbb{C}}$ and respects the Cartan decomposition. Conversely, given a holomorphic representation of a complex reductive group $U^{\mathbb{C}}$, there exists a Cartan involution θ' of $\mathrm{GL}(V)$ which contains the compact group $\rho(U)$ in its fixed point set.

We may assume that U acts by unitary operators on V by choosing a U-invariant hermitian inner product $\langle\cdot,\cdot\rangle$ on V. Let $f\colon V \to \mathbb{R}$, $f(v) = \frac{1}{2}\|v\|^2 := \frac{1}{2}\langle v, v\rangle$. Then f is a U-invariant strictly plurisubharmonic exhaustion function on V and we get an induced U-invariant Kähler form $\omega := 2\mathrm{i}\partial\overline{\partial} f$. The U-action on V admits a moment map, i. e. a U-equivariant map $\mu\colon V \to \mathfrak{u}^*$ with $d\mu^\xi = \iota_{\xi_V}\omega$. Here U acts on $\mathfrak{u}^*$ by the coadjoint action, μ^ξ is by definition the function $\mu^\xi(v) = \mu(v)(\xi)$, the fundamental vector field ξ_V is given by $\xi_V(v) = \frac{\partial}{\partial t}\big|_0 \rho(\exp(t\xi))v$ and ι_{ξ_V} is the contraction of ω with ξ_V. Explicitly, a moment map is given by

$$\mu^\xi(v) := \frac{\partial}{\partial t}\bigg|_0 f(\exp(\mathrm{i}t\xi) \cdot v) = \mathrm{i}\langle \xi_V(v), v\rangle,$$

where we identify T_vV with V.

Identifying $\mathfrak{u}^*$ and $\mathrm{i}\mathfrak{u}$ with respect to a U-invariant inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{u}^{\mathbb{C}}$, the moment map induces by restriction a map $\mu_{\mathfrak{p}}\colon V \to \mathfrak{p}$, given by $\langle \mu_{\mathfrak{p}}(v), \xi\rangle := \mu_{\mathfrak{p}}^\xi(v) := \mu^{-\mathrm{i}\xi}(v)$. Then $\mu_{\mathfrak{p}}$ is K-equivariant and satisfies the defining equation $\operatorname{grad} \mu_{\mathfrak{p}}^\xi = \xi_V$ where the gradient is taken with respect to the Riemannian metric on V associated to the Kähler metric. Therefore we call $\mu_{\mathfrak{p}}$ a *G-gradient map.*

We call a G-invariant locally closed subset X of V a *G-semistable space* if each G-orbit which is closed in X is also closed in V. In the following we consider the restriction $\mu_{\mathfrak{p}}\colon X \to \mathfrak{p}$ of the gradient map to X. Define $\mathcal{M}_{\mathfrak{p}}$ to be the zero fiber $\mu_{\mathfrak{p}}^{-1}(0) \subset X$.

The following results are established in [RS90] and [HS07b].

Theorem 2.1. *The topological Hilbert quotient $\pi\colon X \to X/\!\!/G$ exists and has the following properties.*

(1) *Each fiber of π contains a unique closed G-orbit.*

(2) *Each non-closed orbit in a fiber of π has strictly larger dimension than the closed orbit in that fiber and contains the closed orbit in its closure.*
(3) *For $x \in \mathcal{M}_\mathfrak{p}$ the orbit $G \cdot x$ is the unique closed orbit in the fiber $\pi^{-1}(\pi(x))$.*
(4) *The inclusion $\mathcal{M}_\mathfrak{p} \hookrightarrow X$ induces a homeomorphism $\mathcal{M}_\mathfrak{p}/K \to X/\!\!/G$.*

$$\begin{tikzcd} \mathcal{M}_\mathfrak{p} \arrow[r, hook] \arrow[d] & X \arrow[d] \\ \mathcal{M}_\mathfrak{p}/K \arrow[r, "\sim"] & X/\!\!/G \end{tikzcd}$$

(5) *The restriction $\pi|\mathcal{M}_\mathfrak{p}$ of the quotient map is proper and open.*

In particular, the theorem states that the orbits which intersect $\mathcal{M}_\mathfrak{p}$ are exactly the closed orbits and that $\mathcal{M}_\mathfrak{p}$ intersects each closed orbit in a K-orbit. So the quotient $\mathcal{M}_\mathfrak{p}/K$ parameterizes the closed G-orbits in X.

Remark. Note that in particular the quotient $V \to V/\!\!/G$ exists. Then $X \subset V$ is a G-semistable space if and only if it is G-locally closed in V.

For later use we record

Corollary 2.2. *Let $Y \subset X$ be G-locally closed. Then $\overline{Y \cap \mathcal{M}_\mathfrak{p}} = \overline{Y} \cap \mathcal{M}_\mathfrak{p}$.*

Proof. Let $x \in \overline{Y} \cap \mathcal{M}_\mathfrak{p}$ and let $x_n \in Y$ be a sequence converging to x. Then there exist $y_n \in \overline{G \cdot x_n} \cap \mathcal{M}_\mathfrak{p}$. Note that $y_n \in Y$ since Y is G-locally closed. The sequence $\pi(y_n) = \pi(x_n)$ converges to $\pi(x)$. Since the restriction of π to $\mathcal{M}_\mathfrak{p}$ is proper, we may assume that y_n converges to a $y \in \mathcal{M}_\mathfrak{p}$. Then $y \in \overline{Y}$ and $\pi(y) = \pi(x)$. Since $\pi^{-1}(\pi(x)) \cap \mathcal{M}_\mathfrak{p}$ is a K-orbit we conclude $x \in K \cdot y \subset \overline{Y \cap \mathcal{M}_\mathfrak{p}}$. □

Corollary 2.3. *Let $G \cdot x \subset X$ be a closed orbit. Then every G-invariant open neighborhood of $G \cdot x$ contains a G-open neighborhood.*

Proof. Let W be a G-invariant open neighborhood of $G \cdot x$. The set $\pi^{-1}(\pi(W \cap \mathcal{M}_\mathfrak{p}))$ is G-open, contains $G \cdot x$ and is contained in W. □

If H is a closed compatible subgroup of G then it follows from the definition that a G-semistable space is also an H-semistable space. Compatible subgroups of G occur naturally as isotropy groups of closed orbits.

Lemma 2.4. *(Lemma 5.5 in [HS07b]) Let X be a G-semistable space and let $x \in \mathcal{M}_\mathfrak{p}$. Then the isotropy group G_x is a closed compatible subgroup of G with Cartan decomposition $G_x = K_x \exp(\mathfrak{p}_x)$ where $\mathfrak{p}_x := \{\xi \in \mathfrak{p};\ \xi_X(x) = 0\}$.*

2.4. **The Slice Theorem.** Let $X \subset V$ be a G-semistable space. From [HS07b] we recall the construction of a slice at a point $x \in \mathcal{M}_\mathfrak{p}$ in the case $X = V$. First, the action of G on X induces an isotropy representation of G_x on the tangent space T_xX. Then T_xX can be G_x-equivariantly identified with X. Note that T_xX is a G_x-semistable space since G_x is a compatible subgroup of G. The tangent space $T_x(G \cdot x)$ of the orbit $G \cdot x$ is a G_x-invariant subspace of T_xX.

Lemma 2.5 (Corollary 14.9 in [HS07b])**.** *The G-representation V is completely reducible.*

Since G_x is compatible, $V \cong T_xX$ is completely reducible as a G_x-representation. Therefore there exists a G_x-invariant subspace W of T_xX such that $T_xX = T_x(G \cdot x) \oplus W$. The isotropy G_x acts on the product $G \times W$ by $h \cdot (g, s) = (gh^{-1}, hs)$. We denote the quotient of this action by $G \times^{G_x} W$ and we denote the element $G_x \cdot (g, s)$ by $[g, s]$. The action of G on X induces a map $G \times^{G_x} W \to GW \subset X$. There exists a G_x-invariant neighborhood S of 0 in W such that

the restriction $G \times^{G_x} S \to GS$ is a diffeomorphism onto an open subset GS of X. Moreover, by Corollary 2.3, S can be chosen such that GS is G-saturated in X and S is G_x-open in T_xX. In particular, S is a G_x-semistable space.

Similarly, for an arbitrary G-semistable space X we get ([St08], Corollary 4.5, Lemma 4.9)

Theorem 2.6 (Slice Theorem)**.** *Let X be a G-semistable space and let $x \in \mathcal{M}_{\mathfrak{p}}$. Then there exists a locally closed G_x-stable subset S of X containing x, such that GS is G-open in X and such that the map*

$$\Psi\colon G \times^{G_x} S \to GS, \quad \Psi([g, s]) = gs$$

is a homeomorphism. The slice S is G_x-equivariantly homeomorphic to a G_x-locally closed subset of $T_xV \cong V$ which contains 0.

With the same notation as in the Slice Theorem, we call the data (G_x, S, V) a slice model at x. We will identify $S \subset X$ with the corresponding G_x-locally closed subset of V without further mentioning it. Let (G_x, S, V) be a slice model at $x \in \mathcal{M}_{\mathfrak{p}}$ and $g \in G$. Then $G_{gx} = gG_xg^{-1}$ and $g \cdot S$ contains gx. Then we get a homeomorphism $G \times^{G_{gx}} gS \to GS$. This shows that the assumption $x \in \mathcal{M}_{\mathfrak{p}}$ could be replaced by the assumption that $G \cdot x$ is closed. Note that G_{gx} is not necessarily a compatible subgroup of G.

2.5. **Locally semistable spaces.** Let X be a topological G-space such that the topological Hilbert quotient $\pi\colon X \to X/\!\!/G$ exists. We call X a *locally G-semistable space* if every $x \in X$ has a G-open neighborhood W which admits the structure of a G-semistable space, i. e. there exists a complex reductive group $U^{\mathbb{C}}$ containing G as a compatible subgroup and a holomorphic $U^{\mathbb{C}}$-representation space such that W is G-equivariantly homeomorphic to a G-locally closed subset of V.

Example 2.7. Let X be an affine complex variety equipped with a regular action of a complex reductive group G such that the algebraic Hilbert quotient exists. Then the topological Hilbert quotient exists and topologically coincides with the algebraic quotient. Moreover, X can be G-equivariantly embedded into a regular G-representation, so X is a G-semistable space. More generally, if X is an arbitrary complex variety, such that the good quotient (see [BCM02]) exists, then X is a locally G-semistable space.

Similarly, a complex space with a holomorphic action of a complex reductive group G such that the analytic Hilbert quotient exists, is a locally G-semistable space. Here this follows from [Sn82].

Lemma 2.8. *Let X be a locally G-semistable space and let H be a closed compatible subgroup of G. Then X is a locally H-semistable space.*

Proof. If X is a G-semistable space, then it follows from the definition that it is also an H-semistable space. In particular, the quotient with respect to the action of H exists. Since a locally G-semistable space X is covered by G-open subsets which are G-semistable spaces, it follows that the quotient $X \to X/\!\!/H$ exists. Since a G-saturated subset is also H-saturated, the claim follows. □

The notion of the zero fiber of a gradient map does not make sense for a locally G-semistable space. As a substitute, we define $X_{cc} \subset X$ to be the set of points $x \in X$ such that $G \cdot x$ is closed and such that G_x is compatible. For a G-semistable space we have $\mathcal{M}_{\mathfrak{p}} \subset X_{cc}$. In particular, in a locally G-semistable space each closed G-orbit intersects X_{cc} since it has a neighborhood which is a G-semistable space. Conversely for a locally G-semistable space X and an $x \in X_{cc}$ there exists a G-saturated open neighborhood of x which is a G-semistable space such that $x \in \mathcal{M}_{\mathfrak{p}}$. For this, we first observe that there exists a slice model (G_x, S, V) at $x \in X_{cc}$ since by definition there exists a G-open neighborhood of x which admits the structure of a G-semistable space

containing $G \cdot x$ as a closed orbit. In particular the G-action on this neighborhood is given by restriction of a holomorphic $U^{\mathbb{C}}$-representation, where $U^{\mathbb{C}}$ is complex reductive and contains G as a compatible subgroup. The complex analytic Zariski closure $\overline{G}_x^Z$ of G_x in $U^{\mathbb{C}}$ is compatible, contains G_x and is contained in $(U^{\mathbb{C}})_x$. Therefore $G \cap \overline{G}_x^Z = G_x$. Since V is a holomorphic $\overline{G}_x^Z$-representation space, we get an embedding $G \times^{G_x} S \hookrightarrow U^{\mathbb{C}} \times^{\overline{G}_x^Z} V$. By Lemma 1.16 in [Sj95], there exists a proper holomorphic $U^{\mathbb{C}}$-equivariant embedding of $U^{\mathbb{C}} \times^{\overline{G}_x^Z} V$ into a holomorphic $U^{\mathbb{C}}$-representation space such that $[e, 0]$ has minimal distance to 0. Then it follows from the definition of the gradient map that $[e, 0] \in \mathcal{M}_{\mathfrak{p}}$. Moreover, $G \times^{G_x} S$ is G-locally closed in the holomorphic $U^{\mathbb{C}}$-representation. This shows that the slice neighborhood GS is a G-open neighborhood of x which admits the structure of a G-semistable space such that $x \in \mathcal{M}_{\mathfrak{p}}$.

Moreover, we observe that a topological G-space X with topological Hilbert quotient $\pi \colon X \to X/\!\!/G$ is a locally G-semistable space if and only if for each $x \in X$ there exists a complex reductive group $U^{\mathbb{C}}$ containing G as a compatible subgroup and a G-open neighborhood of x in X which is G-equivariantly homeomorphic to $G \times^H S$ where H is a compatible subgroup of G such that $G \cap \overline{H}^Z = H$ and such that S is H-locally closed in a holomorphic $\overline{H}^Z$-representation space V. Here the Zariski closure of H is taken in $U^{\mathbb{C}}$.

Example 2.9. Let X be a smooth G-manifold and assume that the action of G is proper. Then the topological Hilbert quotient coincides with the geometric quotient and at each $x \in X$ there exists a slice $G \times^{G_x} S \to GS$ where S is an open neighborhood of 0 in a G_x-representation space V by [Pa61]. The isotropy G_x is compact, so the G_x-representation V extends to a $(G_x)^{\mathbb{C}}$-representation $V^{\mathbb{C}}$. Then X is a locally G-semistable space since $(G_x)^{\mathbb{C}} = \overline{G}_x^Z$ where the Zariski closure is taken in $(U_G)^{\mathbb{C}}$.

In a G-semistable space, a closed G-orbit intersects $\mathcal{M}_{\mathfrak{p}}$ in a unique K-orbit. In order to describe the intersection $G \cdot x \cap X_{cc}$, we need the following two lemmas.

Lemma 2.10. *Let H be a compatible subgroup of G. Then the normalizer $\mathcal{N}_G(H)$ of H in G is compatible.*

Proof. The normalizer $\mathcal{N}_G(H)$ is invariant under the Cartan involution θ, so it suffices to show that is consists only of finitely many connected components. By [Po98], the group $H \cdot \mathcal{Z}_G(H)$, where $\mathcal{Z}_G(H)$ denotes the centralizer of H in G, is of finite index in $\mathcal{N}_G(H)$. Let $U^{\mathbb{C}}$ be a complex reductive group containing G as a compatible subgroup. Then $\mathcal{Z}_G(H) = \mathcal{Z}_G(\overline{H}^Z) = G \cap \mathcal{Z}_{\overline{G}^Z}(\overline{H}^Z)$. The centralizer $\mathcal{Z}_{\overline{G}^Z}(\overline{H}^Z)$ is invariant under the Cartan involution θ and it has only finitely many connected components since it is an algebraic group. Therefore it is compatible. Then $\mathcal{Z}_G(H)$ is compatible since the intersection of two compatible subgroups is compatible. Thus $H \cdot \mathcal{Z}_G(H)$ consists of only finitely many connected components and the claim follows. □

Lemma 2.11. *Let H be a compatible subgroup of G. Let $g \in G$. Then gHg^{-1} is compatible if and only if $g \in K \cdot \mathcal{N}_G(H)$. In particular, if gHg^{-1} is compatible, then gHg^{-1} and H are conjugate in K, so $gHg^{-1} = kHk^{-1}$ for some $k \in K$.*

Proof. Let $H = L \exp(\mathfrak{q})$ be the Cartan decomposition of H. Then for $k \in K$, we have $kHk^{-1} = kLk^{-1} \exp(\mathrm{Ad}(k)\mathfrak{q})$. Since the adjoint action of K stabilizes $\mathfrak{p}$, we conclude that kHk^{-1} is compatible. So for $g \in K \cdot \mathcal{N}_G(H)$, the group gHg^{-1} is compatible.

Conversely, if gHg^{-1} is compatible, we may assume $g = \exp(\xi)$ where $\xi \in \mathfrak{p}$. The groups H and gHg^{-1} are θ-stable since they are compatible. Therefore $gHg^{-1} = \theta(gHg^{-1}) = \theta(g)H\theta(g^{-1})$. Explicitly, we have $\exp(\xi)H\exp(-\xi) = \exp(-\xi)H\exp(\xi)$ or equivalently $\exp(2\xi) \in \mathcal{N}_G(H)$.

Since $\mathcal{N}_G(H)$ is compatible by Lemma 2.10 and since $\xi \in \mathfrak{p}$, we conclude that ξ is contained in the Lie algebra of $\mathcal{N}_G(H)$ which implies $g \in \mathcal{N}_G(H)$. □

It follows from Lemma 2.11 that for $x \in X_{cc}$ the orbit $G \cdot x$ intersects X_{cc} in $K \cdot \mathcal{N}_G(G_x) \cdot x$. In particular, for $y \in G \cdot x \cap X_{cc}$ the isotropy G_y is conjugate to G_x in K.

3. Isotropy Stratification

3.1. **The Isotropy Stratification Theorem.** Let X be a locally G-semistable space and let $\pi\colon X \to X/\!\!/G$ be the topological Hilbert quotient. For a closed compatible subgroup H of G we define

$$X^{<H>} := \{x \in X_{cc};\, G_x = H\}.$$

Note that we have a partition $X_{cc} = \bigcup_H X^{<H>}$, where the union is taken over all compatible isotropy groups of points on closed orbits. We call the G-saturated set

$$I_H(X) := \pi^{-1}(\pi(X^{<H>})) = \{x \in X;\, \overline{G \cdot x} \cap X^{<H>} \neq \emptyset\}$$

the *G-isotropy stratum of H in X*. In other words, $I_H(X)$ consists of those fibers of π, where the unique closed orbit is of orbit type G/H. We abbreviate $I_H(X)$ by I_H and call I_H a *stratum*, if no confusion is possible.

Theorem 3.1 (Isotropy Stratification Theorem)**.** *Let X be a locally G-semistable space. Let $\mathcal{I}$ be an index set and let H_i, $i \in \mathcal{I}$, be compatible subgroups of G such that $\{H_i;\, i \in \mathcal{I}\}$ is a set of representatives of conjugacy classes of isotropy groups of closed G-orbits in X. Then*

(1) *I_{H_i} is G-saturated and locally closed.*
(2) *$X = \bigcup_{i \in \mathcal{I}} I_{H_i}$ and the union is disjoint and locally finite.*
(3) *If $\overline{I_{H_i}} \cap I_{H_j} \neq \emptyset$ and $I_{H_i} \neq I_{H_j}$ then there exists a $g \in G$ with $gH_ig^{-1} \subsetneqq H_j$.*

Example 3.2. Consider the adjoint representation of a connected semisimple group G. There exist finitely many θ-stable Cartan subalgebras $\mathfrak{h}_1, \ldots, \mathfrak{h}_n$ in the Lie algebra $\mathfrak{g}$ of G such that each Cartan subalgebra is conjugate to one of these. A G-orbit in $\mathfrak{g}$ is closed if and only if it intersects a Cartan subalgebra. If $\xi \in \mathfrak{h}_i$ is a regular element, then the isotropy G_ξ equals the centralizer $H_i := Z_G(\mathfrak{h}_i)$ of $\mathfrak{h}_i$ in G. Since a neighborhood of ξ in $\mathfrak{h}_i$ is a slice for the G-action, we conclude that the strata $I_{H_1}, \ldots, I_{H_n}$ are the open strata in $\mathfrak{g}$.

In the special case where G is complex reductive, all Cartan subalgebras are conjugate to each other, hence there exists an open and dense stratum.

3.2. **The proof of Theorem 3.1.** First note that I_{H_i} is G-saturated by definition and that X is the union of the strata since each orbit contains a closed orbit in its closure and since each closed orbit intersects X_{cc}. The union is disjoint for if two strata I_H and $I_{H'}$ intersect, the intersection contains a closed orbit, which implies that H and H' are conjugate in G.

Locally the stratification is determined by the stratification of a slice. For this, let (G_x, S, V) be a slice model at $x \in X_{cc}$. Since S is a G_x-semistable space, we have the notion of G_x-isotropy strata in S. Moreover, a G-orbit $G \cdot y$ with $y \in S$ is closed in X if and only if the G_x-orbit $G_x \cdot y$ is closed in S. Note also that $(G_x)_y = G_y$. Identifying GS with $G \times^{G_x} S$, we get

$$I_H(X) \cap GS = \bigcup_{H_i} G \times^{G_x} I_{H_i}(S),$$

for a compatible subgroup H of G. Here the union is taken over all subgroups H_i of G_x such that H_i is conjugate to H in G. In particular, we obtain $I_{G_x}(X) \cap GS = G \cdot I_{G_x}(S)$. The stratum $I_{G_x}(S)$ is given by the intersection of S with $I_{G_x}(V)$. But the closed G_x-orbits in V with isotropy conjugate to G_x are the G_x-fixed points V^{G_x} and the orbits which contain a fixed point in their closure are orbits through points $v \in V$ which are the sum of a fixed point and an

element of the nullcone $\mathcal{N} := \{v \in V; 0 \in \overline{G_x \cdot v}\}$. This implies $I_{G_x}(S) = S \cap (V^{G_x} + \mathcal{N})$, where $V^{G_x} + \mathcal{N} := \{v_1 + v_2; v_1 \in V^{G_x}, v_2 \in \mathcal{N}\}$. Since the nullcone is real algebraic in V (Lemma 7.1 in [HS07a]), this shows that $I_{G_x}(S)$ is locally closed which in turn implies that $I_{G_x}(X)$ is locally closed.

We show that the stratification is locally finite. If X is a G-representation space, the strata are cones which by definition means that they are invariant under multiplication with positive real numbers. Then the stratification is locally finite at $0 \in X$ if and only if there exist only finitely many strata in X. An arbitrary locally G-semistable space is covered by slice neighborhoods and the strata in a slice neighborhood are determined by the strata in a slice. A slice is a subspace of a representation space, so local finiteness follows from

Proposition 3.3. *Assume X is a G-representation which is given by restriction of a holomorphic $U^{\mathbb{C}}$-representation. Then there exist only finitely many strata in X.*

Proof. Since the representation is completely reducible (Lemma 2.5), we have an invariant decomposition $X = W \oplus X^G$. Replacing X by W, we may assume $X^G = \{0\}$.

Recall that we have the notion of a gradient map and its zero fiber $\mathcal{M}_{\mathfrak{p}}$ on the G-semistable space X. The strata in X are determined by the orbit types of closed orbits in X and the closed orbits intersect $\mathcal{M}_{\mathfrak{p}}$. But the strata as well as $\mathcal{M}_{\mathfrak{p}}$ are cones in X, so up to the nullcone $I_G(X)$ every stratum intersects $S^n \cap \mathcal{M}_{\mathfrak{p}}$ where S^n is a sphere which is defined with respect to an arbitrary inner product. Therefore it suffices to show that $\mathcal{M}_{\mathfrak{p}} \cap S^n$ intersects only finitely many strata. Let (G_v, S, V) be a slice model at $v \in \mathcal{M}_{\mathfrak{p}} \cap S^n$. Since the compact set $\mathcal{M}_{\mathfrak{p}} \cap S^n$ is covered by finitely many slice neighborhoods, it suffices to show that the slice neighborhood GS consists of only finitely many strata. But the G-isotropy strata in GS are determined by the G_v-isotropy strata in S which in turn are determined by the G_v-isotropy strata in the representation space V. Moreover G_v is a proper compatible subgroup of G since $X^G = \{0\}$. Therefore the proposition follows by induction over the dimension and the number of connected components of G. □

Remark. If G acts properly on V, then there exists an open and dense stratum in V. The proof is similar to that of Proposition 3.3. Since every G-orbit is closed, the nullcone is trivial and the sphere S^n can be covered by finitely many slice neighborhoods where each slice neighborhood contains a dense stratum by induction. If the dimension of V is greater than one, then the claim follows since S^n is connected. If the dimension of V equals one, then there is a dense stratum since the strata are cones.

More generally, if X is a smooth G-manifold such that the G-action is proper and if the quotient X/G is connected, then there exists a dense stratum in X.

It remains to show the last statement of Theorem 3.1. For this, let $x \in \overline{I_{H_i}} \cap I_{H_j}$. Since the intersection is G-saturated, we may assume that $G \cdot x$ is closed and that $G_x = H_j$. Let (H_j, S, V) be a slice model at x. The slice neighborhood GS and $\overline{I_{H_i}}$ are G-saturated, so GS intersects $X^{<H_i>}$. But this implies that H_i is conjugate to a subgroup of H_j. Thus the proof of Theorem 3.1 is completed.

3.3. **Strata and Slices.** Let $x \in X_{cc}$ and let (G_x, S, V) be a slice model at x. We will give a criterion for which choice of H_i the stratum $I_{H_i}(S)$ in S is non-empty in Lemma 3.5. For this, we first observe that locally the defining set $X^{<H>}$ of $I_H(X)$ is determined by $S^{<H>} = \{y \in S; (G_x)_y = H, G_x \cdot y \text{ closed}\}$.

Lemma 3.4. *Let (G_x, S, V) be a slice model at $x \in X_{cc}$ and let H be a compatible subgroup of G_x. Then there exists an open $\mathcal{N}_G(H)$-invariant neighborhood W of x in X which contains S such that*

(1) $W \cap X^{<H>} = \mathcal{N}_G(H) \cdot S^{<H>}$ *and*

(2) $W \cap X^H = \mathcal{N}_G(H) \cdot S^H$.

Proof. Let $p\colon G \times^{G_x} S \to G/G_x$ denote the projection $p([g,s]) = g \cdot G_x$. The set of H-fixed points in the homogeneous space G/G_x consists of isolated $\mathcal{N}_G(H)$-orbits. Therefore there exists an open $\mathcal{N}_G(H)$-invariant neighborhood W' of $e \cdot G_x$ in G/G_x such that the H-fixed points in W' are given by $\mathcal{N}_G(H) \cdot e \cdot G_x$. Defining $W := p^{-1}(W')$, the claim follows from G-equivariance of p. $\square$

In the rest of this paper we assume that *every non-empty G_x-isotropy stratum in the slice S contains x in its closure*. This can always be achieved by replacing S by an appropriate open neighborhood of x. More precisely, we remove the closures of the strata which do not contain x in their closure. By Proposition 3.3 this is only a finite number of strata. Then we choose a G_x-open neighborhood of x inside the so obtained G_x-invariant open neighborhood. This is possible by Corollary 2.3.

Lemma 3.5. *Let X be a locally G-semistable space, let $x \in X_{cc}$ and let (G_x, S, V) be a slice model at x. Then a G_x-isotropy stratum $I_H(S)$ in S is non-empty if and only if x is contained in the closure $\overline{X^{<H>}}$ of $X^{<H>}$.*

Proof. First, if $x \in \overline{X^{<H>}}$ then $S^{<H>}$ is non-empty by Lemma 3.4 which implies that $I_H(S)$ is non-empty.

Recall that S is a G_x-semistable space with gradient map $\mu_{\mathfrak{p}_x}\colon S \to \mathfrak{p}_x$. Since x is a G_x-fixed point, the orbit $G_x \cdot x = x$ is closed and a fortiori x is contained in the zero-fiber $\mathcal{M}_{\mathfrak{p}_x}$ of $\mu_{\mathfrak{p}_x}$. If $I_H(S)$ is a non-empty stratum, we have $x \in \overline{I_H(S)}$. Then $x \in \overline{I_H(S) \cap \mathcal{M}_{\mathfrak{p}_x}}$ by Corollary 2.2. Thus there exists a sequence $x_n \in I_H(S) \cap \mathcal{M}_{\mathfrak{p}_x}$ which converges to x. The G_x-isotropy at x_n is compatible and conjugate to H. Therefore there exist $k_n \in K_x$ with $k_n x_n \in S^{<H>} \subset X^{<H>}$ by Lemma 2.11. But $k_n x_n$ converges to the K_x-fixed point x so we get $x \in \overline{X^{<H>}}$. $\square$

3.4. **The splitting number.** In this section we assume that X contains a dense stratum I_H. Let $x \in X_{cc}$ and let (G_x, S, V) be a slice model at x. Recall that we assume that every open G_x-isotropy stratum contains x in its closure. We define $n(x)$ to be the number of open G_x-isotropy strata in S. For an arbitrary point $y \in X$ we define $n(y) := n(x)$ where x is a point in the fiber $\pi^{-1}(\pi(y))$ of the quotient $\pi\colon X \to X/\!\!/G$ which is contained in X_{cc}. We call $n(y)$ the *splitting number at y*.

The splitting number is well defined. In order to see that, let $x, x' \in \pi^{-1}(\pi(y)) \cap X_{cc}$. Then $x' = gx$ for a $g \in G$. If S' is a slice at x' then $S := g^{-1}S'$ is a slice at x and the map $g^{-1}\colon S' \to S$, $s \mapsto g^{-1}s$ sets up a one to one correspondence between open strata in S' and open strata in S. The following proposition implies that $n(x)$ does not depend on the choice of the slice at x.

Proposition 3.6. *Let X be a locally G-semistable space which contains a dense stratum $I_H(X)$. Let (G_x, S, V) be a slice model at $x \in X_{cc}$. Then $I_{H'}(S)$ is a non-empty open stratum in S if and only if $kH'k^{-1} = H$ and $kx \in \overline{X^{<H>}}$ for a $k \in K$. The union of the open strata is dense in S and coincides with $I_H(X) \cap S$.*

Proof. By Lemma 3.5 the stratum $I_{k^{-1}Hk}(S)$ is non-empty if and only if $x \in \overline{X^{<k^{-1}Hk>}}$, or equivalently $kx \in \overline{X^{<H>}}$.

Let $I_{H'}(S)$ be any stratum in S which intersects $I_H(X) \cap S$. Then H' is conjugate to H in G and we get $I_{H'}(S) \subset I_H(X) \cap S$. It follows from Theorem 3.1 (3) that $I_{H'}(S)$ is closed in $I_H(X) \cap S$. The G_x-isotropy stratification of V is finite by Proposition 3.3, which implies that every stratum in $I_H(X) \cap S$ is also open in S. In particular the union of the open strata coincides with the open and dense subset $I_H(X) \cap S$.

Finally, a stratum $I_{H'}(S)$ is open if and only if it is contained in $I_H(X) \cap S$ which is the case if and only if H' is conjugate to H in G and then also in K by Lemma 2.11. $\square$

For $x \in X^{<H>}$, the group kHk^{-1} is a subgroup of $G_x = H$ if and only if $k \in \mathcal{N}_K(H)$. This implies

Corollary 3.7. *Let X be a locally G-semistable space which contains a dense stratum $I_H(X)$. Then $n(x) = 1$ for all $x \in I_H$.*

Example 3.8. By the remark following Proposition 3.3, the splitting number is constantly one if X is a smooth G-space such that G acts properly on X.

If X is a complex space or a complex variety equipped with a regular action of a complex reductive group G, then it follows from Luna's Slice Theorem ([Lu73]) that a stratum is locally closed with respect to the Zariski topology. Then, if X is irreducible, there exists a dense stratum in X. In particular, if X is normal, it is irreducible at each point which implies $n(x) = 1$ for all $x \in X$.

4. The Restriction Theorem

4.1. **The Restriction Theorem.** Let X be a locally G-semistable space containing a dense stratum $I_H(X)$. Then X is a locally $\mathcal{N}_G(H)$-semistable space by Lemma 2.8 which in turn implies that the closed $\mathcal{N}_G(H)$-invariant subset $\overline{X^{<H>}}$ of X is a locally $\mathcal{N}_G(H)$-semistable space. In particular the quotient $\pi_N \colon \overline{X^{<H>}} \to \overline{X^{<H>}} /\!\!/ \mathcal{N}_G(H)$ exists.

We now state and prove our main result.

Restriction Theorem. *Let X be a locally G-semistable space containing a dense G-isotropy stratum I_H. Then the inclusion $\overline{X^{<H>}} \hookrightarrow X$ induces a continuous finite surjective map*

$$\Phi \colon \overline{X^{<H>}} /\!\!/ \mathcal{N}_G(H) \to X /\!\!/ G.$$

For $x \in X$, the number of points in the fiber $\Phi^{-1}(\pi(x))$ is equal to the splitting number $n(x)$. If $x \in \overline{X^{<H>}}$ with $n(x) > 1$, then Φ is not open at $\pi_N(x)$.

Here, by a finite map, we mean a proper map with finite fibers.

Remark. If X is a smooth manifold which is covered by G-open sets which are diffeomorphic to G-locally closed submanifolds of holomorphic representation spaces, then $\overline{X^{<H>}}$ is a closed submanifold of X. For the proof, we refer the reader to Section 5.

Since a finite map is closed and a closed bijective map is open, we get

Corollary 4.1. *Assume that $n(x) = 1$ for all $x \in X$. Then Φ is a homeomorphism.*

For a locally G-semistable space X, a stratum I_H and the closure $\overline{I_H}$ are again locally G-semistable spaces. In particular, if X does not contain a dense stratum, we may apply the Restriction Theorem for $\overline{I_H}$. Moreover, for each stratum I_H, the restriction $\Phi \colon X^{<H>} /\!\!/ \mathcal{N}_G(H) \to I_H /\!\!/ G$ is a homeomorphism by Corollary 3.7.

Example 4.2. Φ is a homeomorphism in the situations considered in Example 3.8.

If X is a smooth G-manifold such that the quotient X/G is connected and if the G-action is proper, then a dense stratum I_H always exists. In particular, if G is compact, the assertion of the Restriction Theorem is equivalent to a result in [Sch80].

For the action of a complex reductive group G on an irreducible normal complex affine variety X, a result similar to our Restriction Theorem was established in [LR79]. Here the map $X^H /\!\!/ \mathcal{N}_G(H) \to X /\!\!/ G$ is considered and it is assumed that $X^H /\!\!/ \mathcal{N}_G(H)$ is irreducible. However, $\overline{X^{<H>}}$ is a union of irreducible components of X^H so the quotients $X^H /\!\!/ \mathcal{N}_G(H)$ and $\overline{X^{<H>}} /\!\!/ \mathcal{N}_G(H)$ coincide under the additional assumption that $X^H /\!\!/ \mathcal{N}_G(H)$ is irreducible. Due to normality of X, the map Φ is an isomorphism of affine varieties.

Example 4.3. Let $n \geq 2$, $G := \mathrm{SL}_{2n}(\mathbb{R})$ and consider the natural action of $J := \mathrm{SO}(n,n)$ on $V := \mathbb{R}^{2n}$. Defining $X := G \times^J V$, a G-orbit $G \cdot [g, v]$ in X is closed if and only if the J-orbit $J \cdot v$ is closed in V. Computations show that each closed J-orbit in V intersects one of the rays $\ell_1 := \mathbb{R}^{\geq 0} \cdot (e_1, 0)$ and $\ell_2 := \mathbb{R}^{\geq 0} \cdot (0, e_1)$. In this notation we identify $\mathbb{R}^{2n}$ and $\mathbb{R}^n \times \mathbb{R}^n$. Therefore the quotient $X/\!\!/G$ is homeomorphic to $\ell_1 \cup \ell_2$ which is homeomorphic to $\mathbb{R}$.

Let $H_1 := J_{(e_1,0)}$ and $H_2 := J_{(0,e_1)}$. The J-representation V consists of three strata, namely the nullcone $I_J(V)$ and the open strata $I_{H_1}(V) = \{(v_1, v_2) \in V; \|v_1\| > \|v_2\|\}$ and $I_{H_2}(V) = \{(v_1, v_2) \in V; \|v_1\| < \|v_2\|\}$. In particular, the splitting number at $[e, 0]$ equals 2. The groups H_1 and H_2 are conjugate in K. Explicitly we have $H_1 = k_0 H_2 k_0^{-1}$ for $k_0 = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$. Defining $H := H_1$, we conclude that the stratum $I_H(X) = I_{H_2}(X)$ is dense in X and we may apply the Restriction Theorem. We get

$$\overline{X^{<H>}} = \{[g, v];\ g \in \mathcal{N}_G(H), v \in \ell_1\} \cup \{[gk_0, v];\ g \in \mathcal{N}_G(H), v \in \ell_2\}.$$

Then the quotient $\overline{X^{<H>}}/\!\!/\mathcal{N}_G(H)$ is homeomorphic to $\{[e, v];\ v \in \ell_1\} \cup \{[k_0, v];\ v \in \ell_2\}$, which is homeomorphic to the disjoint union of two rays. With respect to these identifications, the map Φ is given by $\Phi([e, v]) = v$ for $v \in \ell_1$ and $\Phi([k_0, v]) = v$ for $v \in \ell_2$. We observe that Φ glues the two rays at their boundary points.

4.2. **The proof of the Restriction Theorem.** First note that we may assume that X is a G-semistable space. Then the quotient $X/\!\!/G$ is homeomorphic to $\mathcal{M}_{\mathfrak{p}}/K$ where $\mathcal{M}_{\mathfrak{p}}$ is the zero fiber of the gradient map $\mu_{\mathfrak{p}} \colon X \to \mathfrak{p}$. Moreover the quotient $\overline{X^{<H>}}/\!\!/\mathcal{N}_G(H)$ is homeomorphic to $\mathcal{M}_{\mathfrak{n}_{\mathfrak{p}}}/\mathcal{N}_K(H)$. Here $\mathfrak{n}_{\mathfrak{p}}$ denotes the intersection of $\mathfrak{p}$ with the Lie algebra of $\mathcal{N}_G(H)$ and $\mathcal{M}_{\mathfrak{n}_{\mathfrak{p}}}$ is the zero fiber of the $\mathcal{N}_G(H)$-gradient map $\mu_{\mathfrak{n}_{\mathfrak{p}}} \colon \overline{X^{<H>}} \to \mathfrak{n}_{\mathfrak{p}}$.

The following lemma shows in particular that $\mathcal{M}_{\mathfrak{n}_{\mathfrak{p}}}$ is contained in $\mathcal{M}_{\mathfrak{p}}$ which implies that the map Φ corresponds to a map $\phi \colon \mathcal{M}_{\mathfrak{n}_{\mathfrak{p}}}/\mathcal{N}_K(H) \to \mathcal{M}_{\mathfrak{p}}/K$ such that the diagram

$$\begin{tikzcd}
\mathcal{M}_{\mathfrak{n}_{\mathfrak{p}}}/\mathcal{N}_K(H) \arrow[r, "\phi"] \arrow[d, "\sim"] & \mathcal{M}_{\mathfrak{p}}/K \arrow[d, "\sim"] \\
\overline{X^{<H>}}/\!\!/\mathcal{N}_G(H) \arrow[r, "\Phi"] & X/\!\!/G
\end{tikzcd}$$

commutes.

Lemma 4.4. *We have* $\mathcal{M}_{\mathfrak{n}_{\mathfrak{p}}} = \overline{X^{<H>}} \cap \mathcal{M}_{\mathfrak{p}}$.

Proof. Since $\mathfrak{n}_{\mathfrak{p}}$ is a subspace of $\mathfrak{p}$, the inclusion $\overline{X^{<H>}} \cap \mathcal{M}_{\mathfrak{p}} \subset \mathcal{M}_{\mathfrak{n}_{\mathfrak{p}}}$ follows from the definition of the gradient map.

We show $X^{<H>} \cap \mathcal{M}_{\mathfrak{n}_{\mathfrak{p}}} = X^{<H>} \cap \mathcal{M}_{\mathfrak{p}}$. For this let $x \in X^{<H>} \cap \mathcal{M}_{\mathfrak{n}_{\mathfrak{p}}}$. Since $G \cdot x$ is closed, we have $gx \in \mathcal{M}_{\mathfrak{p}}$ for some $g \in G$. The isotropy groups $G_x = H$ and $G_{gx} = gHg^{-1}$ are compatible, which yields $g = kh \in K \cdot \mathcal{N}_G(H)$ by Lemma 2.11. The K-invariance of $\mathcal{M}_{\mathfrak{p}}$ implies $hx \in \mathcal{M}_{\mathfrak{p}} \subset \mathcal{M}_{\mathfrak{n}_{\mathfrak{p}}}$. But then $h \cdot x \in \mathcal{N}_K(H) \cdot x$ since $\mathcal{N}_G(H) \cdot x$ intersects $\mathcal{M}_{\mathfrak{n}_{\mathfrak{p}}}$ in a unique $\mathcal{N}_K(H)$-orbit. Thus $x \in \mathcal{M}_{\mathfrak{p}}$ follows from the K-invariance of $\mathcal{M}_{\mathfrak{p}}$. With Corollary 2.2 we conclude $\mathcal{M}_{\mathfrak{n}_{\mathfrak{p}}} = \overline{X^{<H>} \cap \mathcal{M}_{\mathfrak{n}_{\mathfrak{p}}}} \subset \overline{X^{<H>}} \cap \mathcal{M}_{\mathfrak{p}}$. $\square$

Surjectivity of ϕ and then also of Φ follows from

Lemma 4.5. *We have* $\mathcal{M}_{\mathfrak{p}} = K \cdot \mathcal{M}_{\mathfrak{n}_{\mathfrak{p}}}$. *In particular* $\mathcal{M}_{\mathfrak{p}} \subset K \cdot \overline{X^{<H>}}$.

Proof. The inclusion $K \cdot \mathcal{M}_{\mathfrak{n}_{\mathfrak{p}}} \subset \mathcal{M}_{\mathfrak{p}}$ follows from K-invariance of $\mathcal{M}_{\mathfrak{p}}$ and Lemma 4.4.

For $x \in \mathcal{M}_{\mathfrak{p}} \cap I_H$, the isotropy group G_x is compatible and conjugate to H. Then it is conjugate to H in K by Lemma 2.11. This shows $\mathcal{M}_{\mathfrak{p}} \cap I_H = K \cdot (\mathcal{M}_{\mathfrak{p}} \cap X^{<H>})$ and $\mathcal{M}_{\mathfrak{p}} \cap I_H \subset K \cdot \mathcal{M}_{\mathfrak{n}_{\mathfrak{p}}}$ follows from Lemma 4.4.

We claim that $\mathcal{M}_{\mathfrak{p}} \cap I_H$ is dense in $\mathcal{M}_{\mathfrak{p}}$. Then the assertion of the lemma follows since $K \cdot \mathcal{M}_{\mathfrak{n}_{\mathfrak{p}}}$ is closed. If this is not the case, then there exist an $x \in \mathcal{M}_{\mathfrak{p}}$ and an open neighborhood W of x in $\mathcal{M}_{\mathfrak{p}}$ which does not intersect I_H. Recall that the restriction $\pi\colon \mathcal{M}_{\mathfrak{p}} \to X/\!\!/G$ of the topological Hilbert quotient is open. Therefore $\pi^{-1}(\pi(W))$ is a G-open neighborhood of x in X which does not intersect I_H. This is a contradiction, since I_H is dense in X. So $\mathcal{M}_{\mathfrak{p}} \cap I_H$ is dense in $\mathcal{M}_{\mathfrak{p}}$. $\square$

For $x \in \mathcal{M}_{\mathfrak{n}_{\mathfrak{p}}}$ the number of points in the fiber $\phi^{-1}(\pi(x))$ equals the number of $\mathcal{N}_K(H)$-orbits in $K \cdot x \cap \mathcal{M}_{\mathfrak{n}_{\mathfrak{p}}}$. Here we describe the 1-1-correspondence between these orbits and the open G_x-isotropy strata in a slice at x explicitly. As a consequence we see that the number of points in the fiber $\phi^{-1}(\pi(x))$ is equal to the splitting number $n(x)$.

Proposition 4.6. *Let $x \in \mathcal{M}_{\mathfrak{n}_{\mathfrak{p}}}$ and let (G_x, S, V) be a slice model at x. Then*

$$\begin{aligned}\Psi\colon (K \cdot x \cap \mathcal{M}_{\mathfrak{n}_{\mathfrak{p}}})/\mathcal{N}_K(H) &\to \{\textit{Non-empty open } G_x\textit{-isotropy strata in } S\},\\ \mathcal{N}_K(H)\cdot k \cdot x &\mapsto I_{k^{-1}Hk}(S)\end{aligned}$$

is well-defined and bijective.

Proof. First, we show that Ψ is well-defined. For $kx \in \mathcal{M}_{\mathfrak{n}_{\mathfrak{p}}} \subset \overline{X^{<H>}}$, the stratum $I_{k^{-1}Hk}(S)$ is non-empty and open by Proposition 3.6. Assume that $\mathcal{N}_K(H)\cdot k_1 x = \mathcal{N}_K(H)\cdot k_2 x \subset K\cdot x \cap \overline{X^{<H>}}$ with $k_1, k_2 \in K$. This is equivalent to $k_1 \in \mathcal{N}_K(H)\cdot k_2 \cdot K_x$ which in turn is equivalent to the condition that $k_1^{-1}Hk_1$ and $k_2^{-1}Hk_2$ are conjugate in K_x. But then $k_1^{-1}Hk_1$ and $k_2^{-1}Hk_2$ define the same G_x-isotropy stratum in S.

For injectivity, assume $\Psi(\mathcal{N}_K(H)\cdot k_1 x) = \Psi(\mathcal{N}_K(H)\cdot k_2 x)$. Then $I_{k_1^{-1}Hk_1}(S) = I_{k_2^{-1}Hk_2}(S)$ and the compatible groups $k_1^{-1}Hk_1$ and $k_2^{-1}Hk_2$ are conjugate in G_x. By Lemma 2.11 they are conjugate in K_x. Thus we get $\mathcal{N}_K(H)\cdot k_1 x = \mathcal{N}_K(H)\cdot k_2 x$.

It remains to show that Ψ is surjective. By Proposition 3.6 a non-empty open stratum is of the form $I_{k^{-1}Hk}(S)$ for some $k \in K$ with $kx \in \overline{X^{<H>}}$. Then $kx \in \mathcal{M}_{\mathfrak{n}_{\mathfrak{p}}}$ by Lemma 4.4 and surjectivity is proved. $\square$

The inclusion $\mathcal{M}_{\mathfrak{n}_{\mathfrak{p}}} \hookrightarrow \mathcal{M}_{\mathfrak{p}}$ is continuous and proper. Since $\mathcal{N}_K(H)$ and K are compact, this implies that ϕ is continuous and proper. Hence, ϕ is finite.

To prove the last assertion of the Restriction Theorem let $x, y \in \mathcal{M}_{\mathfrak{n}_{\mathfrak{p}}}/\mathcal{N}_K(H)$ with $x \neq y$ and $\phi(x) = \phi(y)$. Let W_x and W_y be open neighborhoods of x and y, respectively, such that $W_x \cap W_y = \emptyset$. Assume that ϕ is open at x. Since $\mathcal{M}_{\mathfrak{n}_{\mathfrak{p}}} \cap X^{<H>}$ is dense in $\mathcal{M}_{\mathfrak{n}_{\mathfrak{p}}}$ by Corollary 2.2, there exists a $z \in W_y \cap (X^{<H>} \cap \mathcal{M}_{\mathfrak{n}_{\mathfrak{p}}})/\mathcal{N}_K(H)$ satisfying $\phi(z) \in \phi(W_x)$. But then $\phi(z) \in \phi(W_x) \cap \phi(W_y)$, which is impossible since the restriction $\phi\colon (X^{<H>} \cap \mathcal{M}_{\mathfrak{n}_{\mathfrak{p}}})/\mathcal{N}_K(H) \to (I_H \cap \mathcal{M}_{\mathfrak{p}})/K$ is injective by Corollary 4.1.

5. Smoothness of $\overline{X^{<H>}}$

We assume that X is a smooth locally G-semistable space and that I_H is a dense stratum in X. The purpose of this section is to show that then $\overline{X^{<H>}}$ is smooth.

Theorem 5.1. *Assume X is a smooth locally G-semistable space containing a dense stratum $I_H(X)$. Then the closure $\overline{X^{<H>}}$ of $X^{<H>}$ is open and closed in the fixed point set X^H. In particular, $\overline{X^{<H>}}$ is a closed submanifold of X.*

Proof. First, we reduce the assertion of the theorem to the case, where X is a G-representation space. Assuming that X is a G-semistable space, the quotient $\overline{X^{<H>}}/\!\!/\mathcal{N}_G(H)$ is homeomorphic to $\mathcal{M}_{\mathfrak{n}_\mathfrak{p}}/\mathcal{N}_K(H)$. Moreover, we have $\mathcal{M}_{\mathfrak{n}_\mathfrak{p}} \subset \mathcal{M}_\mathfrak{p}$ by Lemma 4.4. Since $\overline{X^{<H>}}$ and X^H are $\mathcal{N}_G(H)$-invariant, it therefore suffices to show that $\overline{X^{<H>}}$ is open in X^H at a point $x \in \overline{X^{<H>}} \cap \mathcal{M}_\mathfrak{p}$. Let (G_x, S, V) be a slice model at x. Locally near x, we have $X^{<H>} = \mathcal{N}_G(H) \cdot S^{<H>}$ and $X^H = \mathcal{N}_G(H) \cdot S^H$ by Lemma 3.4. Then it suffices to show that $\overline{V^{<H>}} = V^H$ since S is an open neighborhood of 0 in V.

By [St09] there exists a subset $\mathcal{U}$ of V^H which is open with respect to the real algebraic Zariski topology such that $G_x \cdot v$ is closed for $v \in \mathcal{U}$. Furthermore, the set $\mathcal{O} := \{v \in V^H; \dim G_x \cdot v \geq \dim G_x/H\}$ is Zariski open in V^H. The intersection $\mathcal{U} \cap \mathcal{O}$ contains $V^{<H>} = I_H(V) \cap V^H$. If $V^{<H>}$ is not dense in V^H, there exists a stratum $I_{H'}(V)$ such that the intersection $I_{H'}(V) \cap \mathcal{U} \cap \mathcal{O}$ contains an interior point v_0 in V^H. Conjugating H' if necessary, we may assume that H' contains H as an open subgroup. By Proposition 3.6, it now suffices to show that $I_{H'}(V)$ is open in V.

Let (H', S_0, W_0) be a slice model at v_0 and let (H, S_1, W_1) be a slice model at $v_1 \in V^{<H>}$. Then W_0 and W_1 are equivalent as H-representation spaces since $V = T_{v_0}(G \cdot v_0) \oplus W_0 = T_{v_1}(G \cdot v_1) \oplus W_1$ are H-invariant decompositions of V and since since $T_{v_0}(G \cdot v_0)$ and $T_{v_1}(G \cdot v_1)$ are equivalent H-representations. Define $W := W_0 \cong W_1$. We have $W = W^H + \mathcal{N}$ where $\mathcal{N}$ is the H-nullcone in W since $I_H(V)$ is open. For openness of $I_{H'}(V)$ we must show $W = W^{H'} + \mathcal{N}'$ where $\mathcal{N}'$ is the H'-nullcone. But we have $\mathcal{N} = \mathcal{N}'$ since H is open in H'. Since v_0 is an interior point of $I_{H'}(V) \cap V^H$ in V^H, there exists a neighborhood D of 0 in W such that $D \cap W^H \subset W^{H'} + \mathcal{N}$. By algebraicity we get $W^H \subset W^{H'} + \mathcal{N}$. But this implies $W = W^H + \mathcal{N} = W^{H'} + \mathcal{N}'$ and the proof is completed. □

References

[BCM02] A. Białynicki-Birula, J. B. Carrell and W. M. McGovern, *Algebraic quotients. Torus actions and cohomology. The adjoint representation and the adjoint action*, Encyclopaedia of Mathematical Sciences 131, Invariant Theory and Algebraic Transformation Groups, II. Springer-Verlag, Berlin, 2002.

[HS07a] P. Heinzner and P. Schützdeller, *Convexity properties of gradient maps*, arXiv:0710.1152v1 [math.CV], 2007.

[HS07b] P. Heinzner and G. Schwarz, *Cartan decomposition of the moment map*, Math. Ann. **337** (2007), 197–232.

[Ho65] G. Hochschild, *The structure of Lie groups*, Holden-Day Inc., San Francisco, 1965.

[LR79] D. Luna and R. W. Richardson, *A generalization of the Chevalley restriction theorem*, Duke Math. J. **46** (1979), no. 3, 487–496.

[Lu73] D. Luna, *Slices étales*, Bull. Soc. Math. France, Mémoire **33** (1973), 81–105.

[Mie07] C. Miebach, *Geometry of invariant subsets in complex semi-simple Lie groups*, Dissertation, Ruhr-Universität Bochum, 2007.

[Pa61] R. S. Palais, *On the existence of slices for actions of non-compact Lie groups*, Ann. of Math. (2) **73** (1961), 295–323.

[Po98] D. Poguntke, *Normalizers and centralizers of reductive subgroups of almost connected Lie groups*, J. Lie Theory **8** (1998), 211–217.

[RS90] R. W. Richardson and P. J. Slodowy, *Minimum vectors for real reductive algebraic groups*, J. London Math. Soc. (2) **42** (1990), no. 3, 409–429.

[Sj95] R. Sjamaar, *Holomorphic slices, symplectic reduction and multiplicities of representations*, Ann. of Math. (2) **141** (1995), no. 1, 87–129.

[Sn82] D. M. Snow, *Reductive group actions on Stein spaces*, Math. Ann. **259** (1982), no. 1, 79–97.

[St08] H. Stötzel, *Quotients of real reductive group actions related to orbit type strata*, Dissertation, Ruhr-Universität Bochum, 2008.

[St09] H. Stötzel, *Closed orbits of real reductive representations*, in preparation.

[Sch80] G. W. Schwarz, *Lifting smooth homotopies of orbit spaces*, Inst. Hautes Études Sci. Publ. Math. **51** (1980), 37–135.

Fakultät für Mathematik, Ruhr Universität Bochum, Universitätsstrasse 150, D - 44780 Bochum
E-mail address: henrik.stoetzel@rub.de